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New results on computable efficiency and its stability for complex networks

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Abstract

The concept of efficiency in a complex dynamic network plays the role of measuring its ability for the exchange of information and its response for the spread of perturbations in diverse applications. This concept is strongly related to the topological properties of the network. A new framework for the definition of network efficiency is discussed, and some of its consequences are established. We also consider how the efficiency changes as the network structure evolves, by introducing the concept of stability associated to the efficiency of a network as an accurate tool to measure the evolution of a dynamic complex network.

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Keywords: Biological networks; Communication networks; Network efficiency; Stability; Network topology; Evolving networks

1. Introduction

Many relevant properties of metabolic pathways, genetic regulatory networks, protein folding and other biological and technological systems may be described in terms of network properties [1,13]. The study of the structural properties of the underlying network can be very important in the understanding

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of the functions of a complex system as well as the response of complex networks to external factors such as the spreading of a perturbation over the network. Thus, the concept of efficiency in a network is strongly related to its topological properties [8–10]. Some relevant definitions of network efficiency, as the definition introduced in [9], can become inappropriate with regard to some important topological properties such as connectedness. For instance, in the case of trophic webs the way in which connectedness is regarded in the definition of efficiency may be crucial for the understanding of how the disappearance of species is to be modelled. This analysis uncovers the need for imposing certain additional, connectedness-dependent properties which should be frequently satisfied by any efficiency definition. Other examples of biological nature include genetic regulatory networks [6], protein folding [15], trophic webs used by ecologists to quantify the interaction between various species, neural networks [17,20], blood vessel networks [13] or disease transmissions and sexual contacts [11]. The study of the structural properties of a complex network representing a system can give relevant information about its behaviour and allows us to efficiently understand and predict the complex dynamics of the system [3,13,18]. A revival of network modelling has been initiated [1,4,5,8–10,13,17–19], resulting in the introduction and study of new classes of modelling paradigms such as evolving networks. In the study of these new models a very useful technique includes the study of how some characteristics and parameters of the network, such as the characteristic path length or the degree distribution, change as the network evolves. An alternative is based on the definition of the efficiency of a network. A network is regarded in terms of how efficiently the propagation of information on a global and local scale, respectively, is determined. By using the efficiency as a new measure to characterize the network [8], it has been shown that small-worlds are systems that are both globally and locally efficient. Moreover, the description of a network in terms of its efficiency also extends the small-world analysis to unconnected networks and to real systems that are better represented as weighted networks. The study of efficiency of a network is not only interesting in computer and communication networks but also in many other examples of complex networks, since it measures how optimally the dynamics of the network takes place and how its behaviour can change due to some variations in the topology of the network. For example, it is crucial to quantify the stability of a cellular network when it is subject to random errors as a result of mutations, harsh extremal conditions that eliminate metabolites or protein misfolding [7], as in trophic networks it is important to analyse the response of the network to the removal, inclusion or mutation of species in an ecosystem. In [4,10], by using as mathematical measures the global and the local efficiency, the authors investigate the effects of errors and attacks both on the global and the local properties of the network, showing that global efficiency is a better measure than the characteristic path length to describe the response of complex networks to external factors.

Let us give some useful mathematical formalism. A generic network may be represented by a *graph* G, which is formed by a finite set V_G of individuals or people, firms, etc. (called *nodes or vertices*) which are connected in some network relationship (each connection is called an *edge*). We will denote the vertices by $i, j, \ldots$ and for simplicity, we write ij to represent the edge linking i and j. A classic graph is the *complete graph* of n vertices, denoted K_n, which is the graph set with all possible edges $\{i, j\}$. We denote by $\mathscr{G}$ the set of all graphs with a finite number of edges. If we want to describe a graph $G \in \mathscr{G}$, it is enough to give the set of its vertices (denoted by V_G) and the set of its edges (denoted by E_G). Therefore, in the sequel, we will denote a graph G, by $G = (V_G, E_G)$. In other contexts, we can also describe a graph G by the so-called *adjacency matrix* (a_{ij}), which is a symmetric $n \times n$ matrix whose entry a_{ij} is 1 if there is an edge joining vertex i to vertex j, and 0 otherwise. The *degree* of a generic node i is the number $gr(i)$ of edges incident with vertex i. A *path* in a graph G between i and j is a finite sequence of nodes

$i_1, \ldots, i_K$ such that $i_k i_{k+1} \in E_G$ for each k, $1 \leqslant k \leqslant K$, with $i_1 = i$ and $i_K = j$. A graph G is *connected* if there exists a path between every pair of vertices $i, j \in V_G$ ($i \neq j$). If a graph G is not connected then it can be expressed as $G = G_1 \cup G_2 \cup \cdots \cup G_m$, where G_i is a connected subgraph of G which is called a *connected component of* G ($i = 1, \ldots, m$).

2. A framework for network efficiency

As it has been mentioned above, the study of the structural properties of the underlying network can be very important for understanding the functions of a complex system [3], because the connectivity structure of a population may affect its dynamics, a typical example being that of the spreading of epidemics over the network. Recently, the efficiency of a network has been introduced as an appropriate parameter to calibrate the performance of a complex network [8–10]. Despite the fact that there are several alternative ways to give a definition of the efficiency $E(\cdot)$ of a network, all of them embody the idea that, if we consider a network G and we construct G' by adding some edges to G, then G' works *more efficiently* than G and therefore $E(G) \geqslant E(G')$. Therefore, the complete graph K_n maximizes the overall efficiency among all possible graphs and $G_\emptyset = (V_G, \emptyset)$ (the empty graph, with no links) is the network structure that minimizes the efficiency. Let us summarize the basic properties that an efficiency should fulfil.

Definition 2.1. Let $\mathscr{G}$ be the set of all graphs with a finite number of vertices. An *efficiency function E* is a function $E : \mathscr{G} \to [0, 1]$ such that

(E1) $E(G_\emptyset) = 0$,
(E2) $E(K_n) = 1$, for all $n \in \mathbb{N}$,
(E3) $E(G) \leqslant E(G')$ if G' is obtained from G by adding edges,
(E4) $E(\cdot)$ is invariant under isomorphism of $\mathscr{G}$,
(E5) $E(G)$ is computable in polynomial time with respect to the number of vertices of G.

Note that we could give a more general definition of efficiency function without stating the property (E5), but such functions are useless for the applications, since we could not deal with networks having a very large number of vertices.

There are several alternative ways of measuring the performance of networks which have been considered throughout the literature. A first example of these mathematical tools is the so-called *characteristic path length* $L(\cdot)$ of a network. To define $L(\cdot)$ we first need to construct a new symmetric distance matrix (d_{ij}) where $d_{ij} = d_{ij}^G$, known in social networks studies as the number of degrees of separation [12], is the minimal number of edges of a path between i and j (called the *distance* between i and j in the graph G). By definition, $d_{ij} \geqslant 1$ with $d_{ij} = 1$ if a direct edge exists between i and j. The characteristic path length L of graph G is defined as the average of the shortest path lengths between two generic vertices:

$$L(G) = \frac{1}{n(n-1)} \sum_{i,j \in G, i \neq j} d_{ij}. \tag{1}$$

Note that the characteristic path length $L(G)$ measures the typical separation between two generic nodes in the network and therefore it is related to the performance of the network: the higher the path length, the more inefficient the network. Despite the fact that the path length is a good parameter to measure the

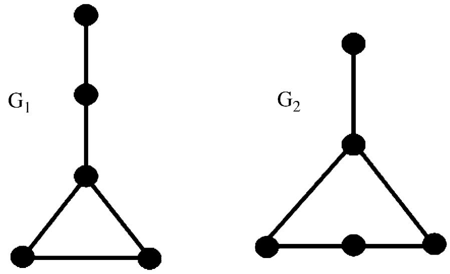

Fig. 1.

(in-)efficiency of a complex network, it is valid only if G is totally connected, which means that there must be at least a path with a finite number of steps connecting any couple of vertices. Otherwise, when we cannot reach j from i, then $d_{ij} = \infty$, and consequently the previous definition of $L(G)$, being divergent, is an ill-defined quantity.

In order to avoid this connection dependence of the characteristic path length, Smith [16], and Latora and Marchiori [8–10] introduced a new definition of the *efficiency* of a network, in order to measure how efficiently the information is exchanged over the network, which also leads to an alternative definition of the small-world behaviour [8,9]. In order to define the efficiency of G, let us suppose that every node sends information along the network through its edges. If we fix two vertices i, j, we measure how efficiently the communication between i and j takes place and we denote by ε_{ij} the two vertices' efficiency. We can assume that the efficiency ε_{ij} in the communication between node i and j is inversely proportional to the shortest distance between such nodes, and therefore it is established that $\varepsilon_{ij} = 1/d_{ij}$. With this definition, when there is no path in the graph between i and j, $d_{ij} = \infty$ and consequently $\varepsilon_{ij} = 0$. So, the efficiency of graph G was defined [4,8–10] as

$$E^+(G) = \frac{1}{n(n-1)} \sum_{i,j \in G, i \neq j} \varepsilon_{ij} = \frac{1}{n(n-1)} \sum_{i,j \in G, i \neq j} \frac{1}{d_{ij}}. \tag{2}$$

It is not difficult to prove that $E^+(\cdot)$ is an efficiency function, since if G' is obtained from G by adding edges, then the number of alternative paths in G' between two vertices i, j is larger than in G and therefore the distance in G' between i, j is smaller than or equal to the distance in G. Note that the definition of $E^+(\cdot)$ only involves the number of vertices of G and how they are connected, so it is not hard to think that $E^+(\cdot)$ must be connected with the number of vertices/edges of the graph. A natural question leads us to speculate about the relationship between the efficiency $E^+(\cdot)$ of a graph and the number of edges or vertices. Intuition seems to suggest that the efficiency is due primarily to the increasing number of edges, but this is not the case. The two graphs in Fig. 1 have the same numbers of vertices and edges (in fact, the same degrees of the vertices), but $E^+(G_1) = \frac{43}{60}$ while $E^+(G_2) = \frac{44}{60}$.

It is not difficult to construct two graphs with the same number of vertices where the one with the fewer edges has greater efficiency. Despite these facts, we can show that there is some relation between the efficiency of a network and the number of vertices and edges, as the following result shows.

Theorem 2.2. *Let G be a graph with* $n > 1$ *vertices and N edges*; *then*

$$\frac{2N}{n(n-1)} \leqslant E^{+}(G) \leqslant \frac{N}{n(n-1)} + \frac{1}{2} \tag{3}$$

and if G is connected, *then*

$$\frac{n-2}{n-1}\frac{2N}{n(n-1)} + \frac{1}{n-1} \leqslant E^{+}(G) \leqslant \frac{N}{n(n-1)} + \frac{1}{2}. \tag{4}$$

Proof. For every vertex i there exist $gr(i)$ vertices whose distance to i is 1, and therefore

$$\begin{aligned}\frac{1}{n(n-1)}\sum_{i=1}^{n} gr(i) \leqslant E^{+}(G) \leqslant \frac{1}{n(n-1)}\sum_{i=1}^{n}\left(gr(i) + (n-1-gr(i))\frac{1}{2}\right)\\ = \frac{1}{n(n-1)}\sum_{i=1}^{n}\left(\frac{gr(i)}{2} + \frac{n-1}{2}\right).\end{aligned}$$

Hence, after Euler's formula

$$\frac{2N}{n(n-1)} \leqslant E^{+}(G) \leqslant \frac{N}{n(n-1)} + \frac{1}{2}.$$

If G is connected, the maximal distance between two vertices is $1/(n-1)$ and thus

$$\frac{1}{n(n-1)}\sum_{i=1}^{n}\left(gr(i) + (n-1-gr(i))\cdot\frac{1}{n-1}\right) \leqslant E^{+}(G),$$

which shows that

$$\frac{n-2}{n-1}\frac{2N}{n(n-1)} + \frac{1}{n-1} \leqslant E^{+}(G) \leqslant \frac{N}{n(n-1)} + \frac{1}{2}. \qquad \square$$

Remark 2.3. The inequalities in the last theorem are sharp, simply by considering $G = K_n$.

We can extend the definition of the efficiency $E^{+}(\cdot)$ and introduce a family of efficiency functions that share the properties of $E^{+}(\cdot)$ by considering other types of averages of ε_{ij}.

Definition 2.4. Let $\alpha \in \mathbb{R}\backslash\{0\}$ and let $G \in \mathcal{G}$. The *α-efficiency* $E^{\alpha}(G)$ is defined by

$$E^{\alpha}(G) = \left(\frac{1}{n(n-1)}\sum_{i,j\in G, i\neq j}\left(\frac{1}{d_{ij}}\right)^{\alpha}\right)^{1/\alpha}. \tag{5}$$

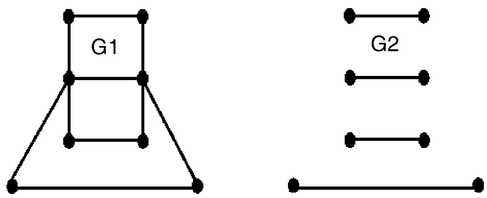

Fig. 2.

Note that $E^1(\cdot) = E^+(\cdot)$ and $E^{-1}(\cdot) = 1/L(\cdot)$. We can prove that $E^\alpha(\cdot)$ is also an efficiency function and verifies some result similar to Theorem 2.2, for all $\alpha > 0$.

The efficiency $E^+(\cdot)$ has been studied and applied to many different problems [8–10], but a simple example (see Fig. 2) will be very useful to illustrate some limitations of the *traditional definition* of network efficiency E^+ (and also of E^α). The main drawback is that the connectedness of the graph does not affect the evaluation of either $E^+(G)$. This may be an inconvenience as in many situations this may not be sensible.

Fig. 2 shows an example of the problems associated with the calculations of $E^+(G)$ and $L(G)$ when the graph is unconnected. We consider two graphs $G1$ and $G2$, both having the same number of nodes, $n = 8$. By using the definition, we obtain that $L(G1) = \frac{19}{14}$ and $L(G2) = \infty$. An alternative possibility to avoid the divergence of $L(G2)$ is to limit the use of expression (1) only to each connected component of $G2$, and then calculate the average. In this way we obtain $L(G2) = 1$ and the final information that we extract from the analysis of the characteristic path length is that graph $G2$ has better structural properties than graph $G1$, since $L(G2) < L(G1)$. This is clearly inexact, because $G1$ is certainly more connected than $G2$, and the misleading information comes from the fact that in the second graph we consider each connected component separately. By studying instead the efficiency of the two graphs we are allowed to take into account in a global analysis all the connected components of both graphs, obtaining $E^+(G1) = \frac{83}{168} = 0,4940$ and $E^+(G2) = \frac{1}{7} = 0,1428$, in perfect agreement with the fact that $G1$ has a much better connectivity than $G2$.

If we wish to avoid this kind of pathology, we should introduce another efficiency function $E^\bullet(\cdot)$ of a network G that is connection sensitive:

Definition 2.5. Let $G \in \mathcal{G}$. The *efficiency* $E^\bullet(G)$ is defined as

$$E^\bullet(G) = \left(\prod_{i,j \in G, i \neq j} \frac{1}{d_{ij}} \right)^{\frac{1}{n(n-1)}}. \tag{6}$$

Note that the new definition $E^\bullet(G)$ represents a *geometric* mean of the two-node distances over G, while $L(G)$ is actually the *arithmetic* mean of the two-node distances in G and $E^+(G)$ is the *harmonic* mean. It is easy to verify that $E^\bullet(\cdot)$ is an efficiency function and $E^\bullet(G)$ vanishes if and only if G is disconnected, i.e., $E^\bullet(\cdot)$ is a connection-sensible parameter that measures how efficiently the information is exchanged over the network and it is more accurate than $1/L(G)$, and for example it can be easily verified that the efficiency of the connected graph $G1$ in Fig. 2 would be $E^\bullet(G1) = 0.3448$, thus smaller than the

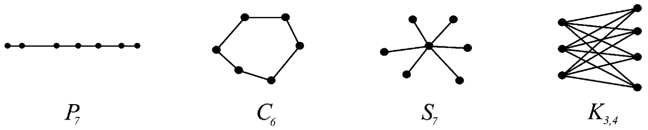

Fig. 3.

traditional value, according to the bounds just established, and of course, $E^{\bullet}(G2) = 0$ according to the requirement of connection sensitivity.

As for $E^{+}(\cdot)$, we can prove (by using the same techniques) that there is some relationship between the efficiency $E^{\bullet}(\cdot)$ of a graph and the number of edges or vertices of the graph, since we obtain the following sharp estimates:

Theorem 2.6. *Let G be a graph with* $n > 1$ *vertices and N edges*; *then*

$$0 \leqslant E^{\bullet}(G) \leqslant \left(\frac{1}{2}\right)^{1-\frac{2N}{n(n-1)}} \tag{7}$$

and if G is connected, *then*

$$\left(\frac{1}{n-1}\right)^{1-\frac{2N}{n(n-1)}} \leqslant E^{\bullet}(G) \leqslant \left(\frac{1}{2}\right)^{1-\frac{2N}{n(n-1)}}. \tag{8}$$

$E^{\bullet}(\cdot)$ is strongly related to $E^{\alpha}(\cdot)$, since $E^{\alpha}(\cdot) \longrightarrow E^{\bullet}(\cdot)$ as $\alpha \to 0$. As a consequence of the arithmetic–geometric mean inequality (AM–GM), we can give the following result that relates the characteristic path length, the classic efficiency, the diameter, the connectedness and the new definition of efficiency $E^{\bullet}(\cdot)$:

Proposition 2.7. *For every* $G \in \mathscr{G}$

$$\frac{1}{\max\{d_{ij};\ \ i \neq j\}} \leqslant \frac{1}{L(G)} \leqslant E^{\bullet}(G) \leqslant E^{+}(G) \leqslant \frac{1}{\min\{d_{ij};\ \ i \neq j\}}. \tag{9}$$

The last result shows that $E^{+}(\cdot)$ is an upper bound for $E^{\bullet}(\cdot)$, but they can behave in different ways, as the following example shows. We consider some classic graphs G_n of n vertices and compute $E^{+}(G_n)$ and $E^{\bullet}(G_n)$ for different values of n to show different behaviours. Let P_n, C_n, S_n and $K_{n,m}$ be the *path*, the *cycle*, the *star* and the *bipartite complete* graphs (see Fig. 3).

Then, we can compute $E^{+}(\cdot)$ and $E^{\bullet}(\cdot)$ and we obtain that for every $n \geqslant 2$

$$E^{+}(P_n) = \frac{2}{n-1}\sum_{i=2}^{n}\frac{1}{i}, \quad E^{\bullet}(P_n) = \left(\prod_{i=1}^{n-1}\left(\frac{1}{i}\right)^{n-i}\right)^{2/(n-1)n},$$

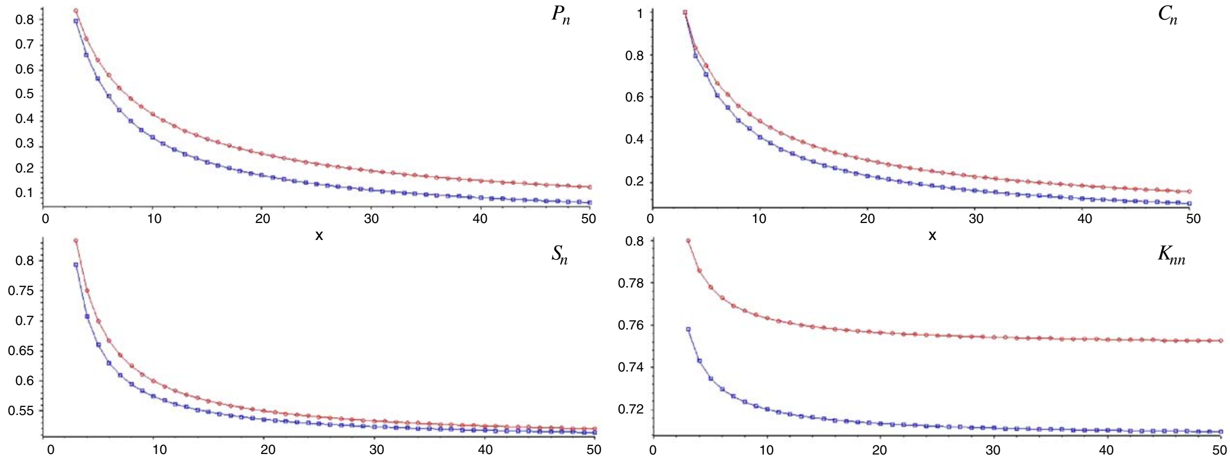

Fig. 4.

$$E^+(C_{2n-1}) = \frac{1}{n-1}\sum_{i=1}^{n}\frac{1}{i}, \quad E^\bullet(C_{2n-1}) = \left(\prod_{i=2}^{n}\frac{1}{i}\right)^{1/(n-1)},$$

$$E^+(C_{2n}) = \frac{2}{2n-1}\left(\frac{1}{2n}+\sum_{i=1}^{n-1}\frac{1}{i}\right), \quad E^\bullet(C_{2n}) = \left(\frac{1}{n}\prod_{i=2}^{n}\frac{1}{i^2}\right)^{1/(2n-1)},$$

$$E^+(S_n) = \frac{1}{2}-\frac{1}{n(n-1)}, \quad E^\bullet(S_n) = \left(\frac{1}{2}\right)^{(n-2)/n},$$

$$E^+(K_{n,m}) = \frac{1}{2}+\frac{nm}{(n+m)(n+m-1)}, \quad E^\bullet(K_{n,m}) = \left(\frac{1}{2}\right)^{\frac{n(n-1)+m(m-1)}{(n+m)(n+m-1)}}.$$

Now, if we compute the E^+ and $E^\bullet$ efficiencies for P_n, C_n, and $K_{n,n}$ for $n=3,\dots,50$ we obtain the results in Fig. 4. Such figure summarizes the graphics of the values of E^+ and $E^\bullet$ for P_n, C_n, and $K_{n,n}$, where $n=3,\dots,50$. In each graphic, we can see two lines (the upper one corresponding to E^+ and the lower one to $E^\bullet$, according to proposition 2.7). A simple analysis of the efficiency functions of P_n and C_n shows that they tend to zero as n goes to $+\infty$, while either E^+ or $E^\bullet$ of the star S_n tends to $\frac{1}{2}$ as n increases. A different phenomenon occurs for the bipartite graph $K_{n,n}$, since in this case $E^+(K_{n,n}) \longrightarrow \frac{3}{4}$ and $E^\bullet(K_{n,n}) \longrightarrow \sqrt{\frac{1}{2}}$, as $n \longrightarrow +\infty$. These facts support the idea that, apart from the connection sensitivity or insensitivity of $E^\bullet$ and E^+, these two efficiency functions behave in a different way, depending on the topology of the network.

We have introduced several alternative efficiency functions and we have verified their basic properties. The next step of this work will lead us to speculate how these efficiency functions behave when we consider two essential transformations/structures in graph theory: subgraphs and products of graphs.

3. Efficiency of subgraphs and products of graphs

3.1. Subgraphs: efficiency of a disconnected network and efficiency of its connected components

If we try to relate the E^+ (or $E^\bullet$) efficiency of a graph G with the efficiency of some subgraph (a graph obtained from G by removing some vertices or edges), it is easy to verify that if we remove some vertices, then it seems that there is no relation between the efficiency of G and the efficiency of the subgraph. If we only remove some edges, by the property (E3) of any efficiency function, the relationship between the efficiency of graphs and subgraphs is straightforward. At midpoint between these two problems (an awkward one: studying the efficiency of general subgraphs, and a trivial one: computing the efficiency of subgraphs with the same vertices) we consider the problem of the relationship between the efficiency of a disconnected graph and the efficiency of its connected components.

A desirable property of an efficiency function is that it is *component sub-additive* i.e., if for every disconnected graph $G = G_1 \cup \cdots G_m \in \mathscr{G}$ (G_i are its connected components),

$$E(G) \geqslant \sum_{i=1}^{m} E(G_i).$$

It is clear that $E^\bullet$ cannot be component sub-additive, since for every disconnected graph $G = G_1 \cup \cdots \cup G_m \in \mathscr{G}$, $E^\bullet(G) = 0$, while $E^\bullet(G_i) \neq 0$. It can be verified that E^+ is not component sub-additive, by taking simple examples, but in the following result we will show that it verifies a weaker condition, since for every disconnected graph $G = G_1 \cup \cdots \cup G_m \in \mathscr{G}$, we prove that

$$E^+(G) \geqslant \sum_{i=1}^{m} c_i E^+(G_i).$$

A problem related with this is to find some relation between the E^+ efficiency of a disconnected graph $G = G_1 \cup \cdots \cup G_m \in \mathscr{G}$ and the $E^\bullet$ efficiency of the connected components. The answers to these two problems are contained in the following result.

Theorem 3.1. *Let $G \in \mathscr{G}$ be a disconnected graph with n vertices such that $G = G_1 \cup \cdots \cup G_m$ (G_i connected component of G with n_i vertices), then*

$$E^+(G) = \sum_{k=1}^{m} \frac{n_k(n_k - 1)}{n(n-1)} E^+(G_k), \tag{10}$$

$$E^+(G) \geqslant \frac{1}{n-1} \prod_{i=1}^{m} (n_i - 1)^{n_1/n} E^\bullet(G_i)^{n_i/n}. \tag{11}$$

Proof. Since G_i is a connected component of G, on the one hand we obtain that

$$E^+(G) = \frac{1}{n(n-1)} \sum_{i \neq j} \left(\frac{1}{d_{ij}} \right) = \frac{1}{n(n-1)} \sum_{k=1}^{m} \sum_{i,j \in G_k} \left(\frac{1}{d_{ij}} \right)$$
$$= \sum_{k=1}^{m} \frac{n_k(n_k-1)}{n(n-1)} E^+(G_k).$$

On the other hand, since

$$E^+(G) = \sum_{k=1}^{m} \frac{n_k(n_k-1)}{n(n-1)} E^+(G_k),$$

by the AM–GM inequality and proposition 2.7, we obtain that

$$E^+(G) = \sum_{k=1}^{m} \frac{n_k}{n} \left(\frac{n_k - 1}{n-1} \right) E^+(G_k) \geqslant \sum_{k=1}^{m} \frac{n_k}{n} \left(\frac{n_k - 1}{n-1} \right) E^\bullet(G_k)$$
$$\geqslant \frac{1}{n-1} \prod_{k=1}^{m} (n_k - 1)^{n_k/n} E^\bullet(G_k)^{n_k/n}. \qquad \square$$

Remark 3.2. Note that (10) and (11) are sharp since if we consider the graph G with 4 vertices and two connected components, then (10) and (11) are equalities.

3.2. Efficiency and product of graphs

Let $G_1, \ldots, G_n$ be graphs with vertices sets $V_1, \ldots, V_n$, respectively. The *graph product* $G_1 \times \cdots \times G_n$ has vertices set $V_1 \times \cdots \times V_n$ with the vertices $u = (g_1, \ldots, g_n)$ and $v = (h_1, \ldots, h_n)$ adjacent if and only if for exactly one i, $g_i \neq h_i$ and $\{g_i, h_i\}$ is an edge in G_i. By P_n, C_n, and K_n, we denote the path, cycle, and complete graph on n vertices, respectively. It is well known that cartesian products like *hypercubes* ($Q_n = (K_2 \times \cdots \times K_2)$), *grids* ($P_{m_1} \times \cdots \times P_{m_n}$), and *tori* ($C_{m_1} \times \cdots \times C_{m_n}$) are highly recommended for the design of interconnection networks in multiprocessor computing systems [2,14].

The problem of the relationship between the efficiency of some networks $G_1, \ldots, G_n \in \mathcal{G}$ and the efficiency of the product $G_1 \times \cdots \times G_n$ is a tough problem, even for the first examples. The next result shows that $E^+(Q_n)$ suffers a drop as $n \longrightarrow +\infty$, causing the efficiency to vanish, and as a consequence of proposition 2.7, also $E^\bullet(Q_n) \longrightarrow 0$.

Proposition 3.3. $E^+(Q_n) \longrightarrow 0$ *as* $n \longrightarrow +\infty$.

Proof. Fix $k > 1$. If $n \geqslant k$, then

$$E^+(Q_n) = \frac{2^n}{2^n(2^n - 1)} \left(\binom{n}{1} \frac{1}{1} + \binom{n}{2} \frac{1}{2} + \cdots + \binom{n}{n} \frac{1}{n} \right).$$

It is not difficult to see that $\binom{n}{1}\frac{1}{1}+\cdots+\binom{n}{k}\frac{1}{k}\leqslant kn^k$. On the other hand,

$$\binom{n}{k+1}\frac{1}{k+1}+\cdots+\binom{n}{n}\frac{1}{n}\leqslant\frac{1}{k+1}\left(\binom{n}{0}+\cdots+\binom{n}{n}\right)=\frac{2^n}{k+1}.$$

This implies that

$$E^+(Q_n)\leqslant\frac{kn^k}{2^n-1}+\frac{2^n}{2^n-1}\frac{1}{k+1}.$$

Thus, for every fixed k, we obtain

$$0\leqslant\limsup_{n\to\infty} E^+(Q_n)\leqslant\frac{1}{k+1},$$

and the result follows. □

4. When the network evolves: from efficiency to stability

While any efficiency function is an indicator of network performance, i.e., of the network capability to have a short-path connection among nodes, it is also necessary to have a measure regarding to what extent can such performance remain relatively unaltered after a perturbation of any kind, let it be a failure or intentional attack, or the opposite, an improvement of the network. In this sense stability is intended to provide a measure of network robustness to perturbations. Therefore stability is independent of the actual value of efficiency (or some other magnitude to be considered instead) but it is related to a relatively small variability of that magnitude when the network experiences a modification.

In what follows, P is going to denote a normalized function $P : G \to [0, 1]$ which is computable in a polynomial order of the time, and which verifies the symmetry property that $P(G) = P(G')$ for any two isomorphic graphs G and G'. Such functions will be called *normal graph functions*. Examples of this type of function are all efficiency functions considered in the last sections. Actually this specific function will be the only case regarded in the rest of the article but more general instances will be envisaged in the future.

Additionally, $O : G \to G'$ will denote an operation transforming a graph $G = (E, V)$ into another one $G' = (E', V')$ and such that G' is obtained from G by removing some vertices/edges or vice versa. This kind of transformation will be termed *graph operation* in what follows. For practical purposes, in the examples considered in this section operation O will always correspond to the suppression of an edge $\{e_i\}$. Such an operation will be termed L_i^-, namely $L_i^-(G) = L_i^-((E, V)) = G' = (E', V')$ where $V' = V$ and $E' = E\backslash\{e_i\}$.

After these preliminary definitions, it is now possible to present the following definition of graph stability:

Definition 4.1. Given a graph G, a graph operation O and a normal graph function P, the *stability* of the graph relative to magnitude P after transformation O is a function $S(G, P, O)$ verifying the following

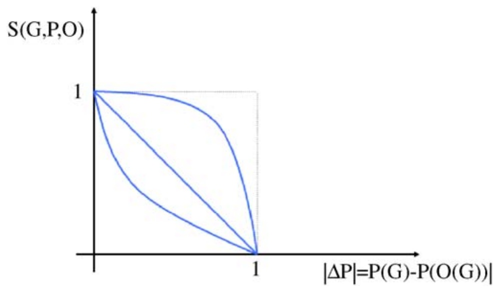

Fig. 5.

properties:

(S1) $S(G, P, O) \in [0, 1]$.
(S2) $S(G, P, O) = 1$ if and only if $P(G) = P(O(G))$.
(S3) $S(G, P, O) = 0$ if and only if $|P(G) - P(O(G))| = 1$.
(S4) $S(G, P, O)$ is a monotonic function of $|P(G) - P(O(G))|$.
(S5) $S(G, P, O) = S(O(G), P, O^{-1})$ where $O^{-1}(O(G)) = G$ for every G.
(S6) $S(G, P, O)$ is computable in polynomial time provided P is.

Consequently, stability is defined as a decreasing (not necessarily strictly) function of $|\Delta P| = |P(G) - P(O(G))|$, as illustrated in Fig. 5.

In what is to follow one of the simplest possible choices in this sense will be the one adopted, namely the linear dependence on $|\Delta P|$, as stated in the following definition:

Definition 4.2. Given a graph G, a graph operation O and a normal graph function P, the *incremental stability* $S_\Delta(G, P, O)$ associated with them is defined as

$$S_\Delta(G, P, O) = 1 - |\Delta P| = 1 - |P(G) - P(O(G))|. \tag{12}$$

We now have:

Proposition 4.3. *The incremental stability is a stability function.*

Proof. The verification of properties (S1)–(S4) and (S6) is straightforward. Regarding property (S5), note that

$$S_\Delta(O(G), P, O^{-1}) = 1 - |P(O(G)) - P(O^{-1}(O(G)))| = S_\Delta(G, P, O). \qquad \square$$

In spite of the linear definition of incremental stability, its actual behaviour is far from trivial even for simple graphs when specific choices are made for P and O. As anticipated, in what is to follow, we shall set $O = L^-$ while P will be one of the efficiencies E^+ or $E^\bullet$. Consider, thus, the dependence with n of $S_\Delta(G, E^+, (L^-)^n)$ and $S_\Delta(G, E^\bullet, (L^-)^n)$, namely the evolution of the incremental stability along the successive suppression of n edges of G, one edge being suppressed at each time. Then the following result holds:

Remark 4.4. $S_\Delta(G, E^+, (L^-)^n)$ and $S_\Delta(G, E^\bullet, (L^-)^n)$ are not necessarily monotonic functions of n.

The results presented in the previous section (Theorems 2.2 and 2.6) allow to give some estimates for $S_\Delta(G, E^+, L^-)$ and $S_\Delta(G, E^\bullet, L^-)$ as the following theorem shows:

Theorem 4.5. *Let $G \in \mathscr{G}$ be a graph with $n > 1$ vertices and N edges; then,*

$$\frac{1}{2} + \frac{N-2}{n(n-1)} \leqslant S_\Delta(G, E^+, L^-) \leqslant 1 - \frac{1}{n(n-1)}. \tag{13}$$

If G is connected then

$$1 - \left(\frac{1}{2}\right)^{1-\frac{2N}{n(n-1)}} \leqslant S_\Delta(G, E^\bullet, L^-) \leqslant 1 - \left(\frac{1}{n-1}\right)^{1-\frac{2(N-1)}{n(n-1)}} \left(1 - \left(\frac{1}{2}\right)^{1/n(n-1)}\right)^2. \tag{14}$$

Proof. The lower bounds for $S_\Delta(G, E^+, L^-)$ and $S_\Delta(G, E^\bullet, L^-)$ in (13) and (14) are a direct consequence of Theorems 2.2 and 2.6, respectively.

If we wish to prove the upper bound for $S_\Delta(G, E^+, L^-)$, note that if we remove the edge ij, then the distance between i and j falls from 1 to $d'_{ij} \geqslant 2$; therefore

$$E^+(G) - E^+(L^-(G)) = \frac{1}{n(n-1)} \sum_{i \neq j} \left(\frac{1}{d_{ij}} - \frac{1}{d'_{ij}}\right) \geqslant \frac{1}{n(n-1)}.$$

Hence, $S_\Delta(G, E^+, L^-) \leqslant 1 - \frac{1}{n(n-1)}$.

By using a similar reasoning, if we remove the edge ij, by using Theorem 2.6, we obtain that

$$\begin{aligned} E^\bullet(G) - E^\bullet(L^-(G)) &= \prod_{k \neq l} \left(\frac{1}{d_{kl}}\right)^{1/n(n-1)} - \prod_{k \neq l} \left(\frac{1}{d'_{kl}}\right)^{1/n(n-1)} \\ &\geqslant E^\bullet(L^-(G)) \left(1 - \left(\frac{1}{2}\right)^{1/n(n-1)}\right)^2 \\ &\geqslant \left(\frac{1}{n-1}\right)^{1-\frac{2(N-1)}{n(n-1)}} \left(1 - \left(\frac{1}{2}\right)^{1/n(n-1)}\right)^2. \end{aligned}$$

Therefore, $S_\Delta(G, E^\bullet, L^-) \leqslant 1 - \left(\frac{1}{n-1}\right)^{1-\frac{2(N-1)}{n(n-1)}} \left(1 - \left(\frac{1}{2}\right)^{1/n(n-1)}\right)^2$. $\square$

Note that the connection hypothesis on G in the last theorem is not critical, since if G is disconnected, then trivially $S_\Delta(G, E^\bullet, L^-) = 1$.

Either Remark 4.4 or Theorem 4.5 are interesting results as they open the door towards future developments on optimization. The best way of verifying the validity of Remark 4.4 is by means of some counterexamples.

In the first example, consider the sequence of graphs H_3 to H_0 appearing as a result of the successive removal of one edge at each step (Fig. 6).

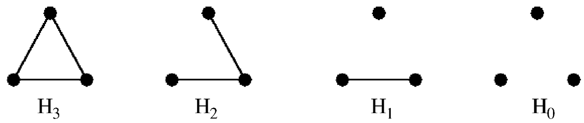

Fig. 6.

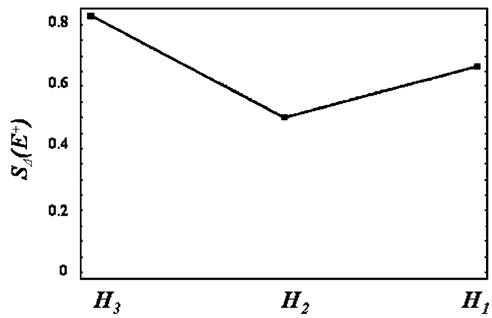

Fig. 7.

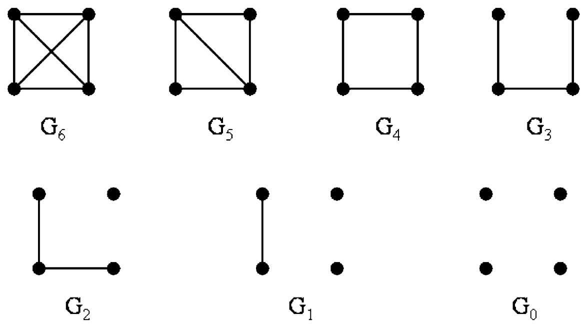

Fig. 8.

The efficiencies and incremental stabilities can be explicitly computed. Therefore if we look at the evolution of the incremental stability S_{Δ} associated with efficiency E^{+}, the result is the one shown in Fig. 7 (notice that S_{Δ} is not defined for H_0 when the graph operation is edge removal).

It is then remarkable that the less efficient graph H_1 is not less stable, which is H_2. In addition, the existence of a global minimum at H_2 demonstrates that the incremental stability may not be a monotonic function of the number of edges suppressed, as anticipated in the previous proposition.

Consider a second example, this time with the sequence of graphs G_6 to G_0, which appears as a result of the iterated application of the same graph operation of one edge removal (Fig. 8).

Now the evolution of the incremental stability for both efficiencies E^{+} and $E^{\bullet}$ is going to be analysed. Let us look first at S_{Δ} for E^{+}. The result is shown in Fig. 9.

In this case the most stable graphs are G_6 and G_5 with exactly the same stability. Therefore, a *"stability plateau"* is present at this region. Then the minimum is reached at G_3 and stability is then increased until

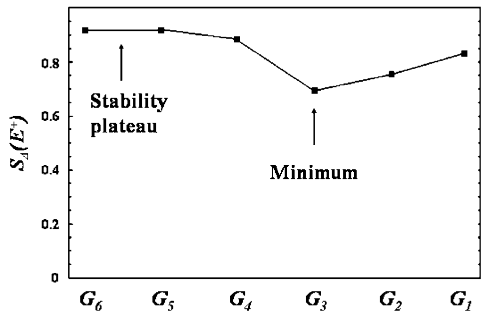

Fig. 9.

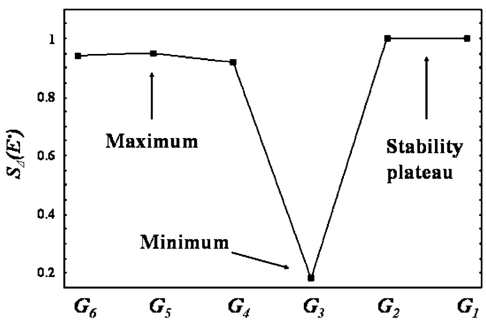

Fig. 10.

G_1. A different result is found, however, in the case of $E^\bullet$ (Fig. 10). Now the most stable connected graph is G_5, which does not coincide with the most efficient graph G_6. On the other hand, stability displays a sharp fall at the minimum G_3. The reason for this is that efficiency $E^\bullet$ falls to zero for disconnected graphs, which is precisely the transition occurring between G_3 and G_2 reflected in the value of S_Δ at G_3. Since G_2, G_1 and G_0 are disconnected their $E^\bullet$ efficiencies are zero and this explains the presence of a stability plateau of maximal value 1 for G_2 and G_1.

5. Conclusions and final remarks

The study of complex networks requires the use of suitable averaged magnitudes allowing a tractable approach to practical issues of description, analysis, design and optimization. In this sense, the notion of efficiency seems natural as it provides a convenient measure applicable to both network functioning and evolution. However, there are still no universally accepted definitions (or even axioms) for such a magnitude, in spite of its already fruitful use in diverse applications. In this sense, it is worth noting that we are facing an open domain in which both an in-depth investigation of the most consistent definition (or definitions) as well as its future use in application-oriented developments regarding design, optimization, and error/attack prevention are still lacking and should deserve future attention.

Stability complements efficiency by giving computable topological measures of network design and robustness. In the second part of this work we have focused on some initial considerations regarding stability, but it is clear that additional possibilities must be considered in order to provide a complete analysis of this parameter. Not only alternative definitions beyond S_Δ are conceivable, but actually normal graph functions different from efficiency (such as vulnerability or cost) may be monitorized from this point of view, and certainly other possible graph operations differing from single-edge removal are sensible. This seems to be an unavoidable consequence of the fact that even a “simple” description of network design in terms of a few complementary parameters naturally leads to a relatively high number of possibilities and perspectives. Probably, this is one of the reasons explaining the remarkable amount of results that applied network topology has produced in the last few years, especially (but not only) with significant perspectives in the domain of network optimization. Such issues deserve future attention from the point of view adopted in this paper.

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